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Competition Versus Optimal Control in Groundwater Pumping

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This article considers one of the most important issues in water resources research, namely, the management of groundwater. Economists have long taken it for granted that the temporal allocation of groundwater would lead to welfare losses if left to the free market because all farmers pump from a common aquifer. Hence water economists studied extensively optimal control of temporal groundwater allocation. They never paused to compare the temporal allocation yielded by optimal control with the free market. In this article we prove by comparing the two strategies analytically that if the storage capacity of the aquifer is relatively large, the difference between them is so small that it can be ignored for practical consideration.

INTRODUCTION

This paper discusses the economic aspects of a model for the problems of farmers pumping groundwater out of an aquifer. This has always been regarded as a typical example of an externality (when all producers increase their output as a whole, the cost function of each individual producer is affected), but this viewpoint needs some elaboration. One could consider the case of pumping an aquifer as lying between fishery harvesting at one extreme and the cutting of privately owned timber at the other. In the case of fisheries the fishing grounds are a nonexclusive resource which consequently leads to dissipation of rent as well as a possibly inefficient distribution of effort over time. In the case of privately owned timber, nonowners are excluded from cutting; hence there is no dissipation of rent, and owners may spread their production over time in an optimal manner. Exclusiveness is present at a very large extent in the aquifer, since only farmers who own land overlying the aquifer can pump water and other farmers are excluded from the resource.

The effect of this exclusiveness was discussed by *Cheung* [1970]. But exclusiveness in the case of farmers using groundwater for irrigation is not as complete as in the case of privately owned timber.

The contrast of the pumping of an aquifer with the cutting of timber revolves about the temporal allocation of groundwater. If each farmer owned his own little aquifer, there would be no problem of externality, and his effort over time would maximize the present value of all future income streams derived from irrigation. This behavior would be identical with that of the timber owner. However, given the usual situation where many farmers pump water from the common aquifer, then the individual farmer cannot expect to have more water in storage for him next year if he pumps less this year. Consequently, instead of maximizing present value, farmers simply pump water each year, satisfying the condition that the marginal cost of pumping equals the value of the marginal physical product (VMP) of water.

The type of externality described above has been discussed in the literature [*Burt*, 1964, 1967, 1970; *Burt and Stauber*, 1971; *Cummings and McFarland*, 1974; *Brown and Deacon*, 1972]. Economists have applied various optimal control (or, equivalently, dynamic programing) methods in order to derive an optimal water use over time. *Brown and Deacon* [1972] derived a formula for a tax that should be imposed on groundwater (pumped) in order to yield the optimal control solution. On the other hand, *Gisser and Mercado* [1973, 1972] studied the temporal allocation of groundwater under competitive conditions (no control), and so the important policy question occurs: Is the difference between the optimal control solution and the competitive solution large enough to justify the use of control? This question is even more relevant when we take into consideration that optimal control is not costless.

Before we embarked on this project we shared the opinion of other economists that an optimal control strategy would yield higher present value of future income streams as compared to no-control strategy. We have selected the Pecos River Basin for our empirical comparison. We simulated the control and no-control (competition) strategies for given estimates of economic and hydrologic parameters. We obtained results that are practically identical for the two strategies. These results, which are of more interest to the operations research profession than to water economists, were published in another journal [*Gisser and Sanchez*, 1980].

In this paper we provide the analytical results which are of more interest to water economists. In what follows we shall proceed as follows. First, we shall describe briefly a model which integrates the demand function for irrigation water with the hydrologic theory, assuming a single-cell aquifer. Second, we shall prove that the difference in temporal allocation yielded by competition, on one hand, and optimal control, on the other, reduces to expressions which incorporate economic and hydrologic parameters that are negligible for all practical purposes when the area times storativity of the aquifer is reasonably large.

THE BASIC MODEL AND COMPETITION (NO CONTROL)

For the benefit of the reader we reproduce the basic elements of the model developed by *Gisser and Mercado* [1973]. Figure 1 illustrates the relationships between the various water flows and the aquifer.

We assume that the demand for irrigation water is a negatively sloped linear function as follows:

$$W = g + kP \tag{1}$$

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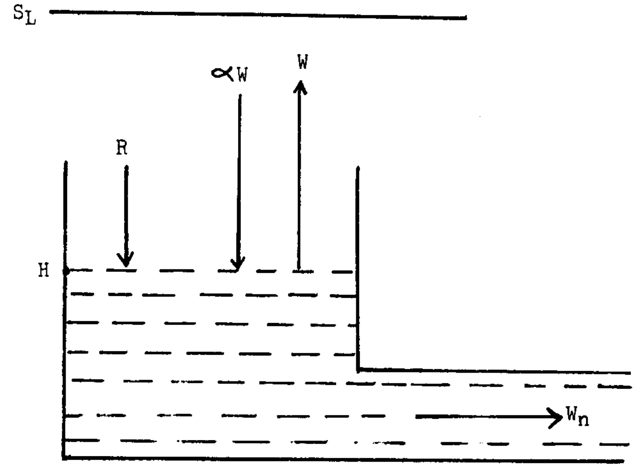

Fig. 1. A model of an aquifer.

where W is pumping measured in acre-feet per unit of time and P is the price of water in dollars.

Following *Martin and Archer* [1971] and *Domenico et al.* [1968], we accept a linear cost function expressed as

$$\bar{P} = C_0' + C_1'(S_L - H) \tag{2}$$

where C_0', which is negligible, represents fixed costs due to the hydrologic cone, C_1' is the marginal pumping cost per acre-foot of water pumped per foot of life, S_L is the elevation of the irrigation surface above sea level, and H is the water table elevation above sea level. $S_L - H$ is thus the total lift in feet. Letting $C_0 = C_0' + C_1'S_L$ and $C_1 = -C_1'$, we have

$$\bar{P} = C_0 + C_1H \tag{3}$$

Equation (1) represents the value of the marginal physical product of irrigation water; this is, accordingly, the demand equation for water. As mentioned above, the commonality aspects of the aquifer imply that farmers equate the value of the marginal physical product of water with the marginal cost of pumping (rather than maximize the present value of all future income streams, which private owners of timber would do). The long-run marginal cost of pumping is $\bar{P}$ in (3). (Notice that if there is any objection to using $\bar{P}$ given in (3) as the long-run marginal cost, C_0', which is negligible, can be set to be equal to zero and the results of the forthcoming analysis would not be changed.) Equating $\bar{P}$, the marginal cost of pumping as given in (3), with P, the value of the marginal physical product of water as given in (1), means that $\bar{P} = P$, and accordingly, if we substitute the right-hand side of (3) for P in (1), we obtain the following equation which governs the behavior of farmers under competition (no control) as follows

$$W = d + kC_1H \tag{4}$$

where

$$d = g + kC_0 \tag{5}$$

The differential equation which describes the water table as a function of time is obtained by equating 'rate in' minus 'rate out' with the impact on the water table, as displayed in Figure 1, namely

$$AS \cdot \dot{H} = R + (\alpha - 1)W \tag{6}$$

where R is natural recharge, α is return flow coefficient, and AS is area of the aquifer times storativity. Substituting the right-hand side of (4) for W in (6), we obtain the differential equation governing the change in the water table over time under competition as follows:

$$AS \cdot \dot{H} = R + (\alpha - 1)d + (\alpha - 1)kC_1H \tag{7}$$

The solution of (7) is

$$H(t) = \frac{-(\alpha - 1)(g + kC_0) - R}{(\alpha - 1)kC_1} + \left\{\frac{R + (\alpha - 1)(g + kC_0) + [(\alpha - 1)kC_1]H_0}{(\alpha - 1)\,kC_1}\right\} \cdot \exp\left\{\frac{[(\alpha - 1)kC_1]t}{AS}\right\} \tag{8}$$

By utilizing (4) and (5) we get the temporal pumping function under competition as follows:

$$W(t) = \frac{-R}{\alpha - 1} + \left\{\frac{R}{\alpha - 1} + g + kC_0 + kC_1H_0\right\} \cdot \exp\left\{\frac{[(\alpha - 1)kC_1]t}{AS}\right\} \tag{9}$$

In the above equations, H_0 is the water level at $t = 0$. Identical results were derived by *Gisser and Mercado* [1973].

OPTIMAL CONTROL

Under the assumption of optimal control, farmers maximize the present values of their future income streams. Total revenue of farmers is the area under the demand curve for irrigation water. It is obtained by expressing P in (1) as an explicit function of W and then integrating from zero to W. Total cost of pumping is obtained by multiplying the cost of pumping per acre-foot of water, as given in (3), by W, the rate of pumping. Thus net farm income per unit of time is total revenue minus total cost:

$$\frac{1}{2k}W^2 - \frac{g}{k}W - (C_0 + C_1H)W \tag{10}$$

The optimal control problem is to maximize

$$\int_0^\infty e^{-rt}\left[\frac{1}{2k}W^2 - \frac{g}{k}W - (C_0 + C_1H)W\right]dt \tag{11}$$

where $r > 0$ is the discount factor and subject to

$$\dot{H} = [R + (\alpha - 1)W]/AS \qquad H(0) = H \tag{12}$$

Notice that (12) is a different representation of (6). To solve this problem, we will use the Pontryagin principle, as discussed for instance, in the books by *Berkovitz* [1974], *Bryson and Ho* [1969], and *Pontryagin et al.* [1962]. Let the Hamiltonian be denoted by $\mathcal{H}$ and write it as

$$\mathcal{H} = -e^{-rt}\left[\frac{1}{2k}W^2 - \frac{g}{k}W - (C_0 + C_1H)W\right] + \lambda[R + (\alpha - 1)W]/AS \tag{13}$$

and then the following equations are satisfied:

$$\frac{\partial\mathcal{H}}{\partial W} = -e^{-rt}\left(\frac{1}{k}W - \frac{g}{k}W - C_0 - C_1H\right) + \frac{\lambda(\alpha - 1)}{AS} = 0 \tag{14}$$

$$\dot{\lambda} = \frac{-\partial\mathcal{H}}{\partial H} = -e^{-rt}C_1W \tag{15}$$

From (14) we can get an expression for λ:

$$\lambda = \frac{AS}{\alpha - 1} e^{-rt} \left(\frac{1}{k} W - \frac{g}{k} - C_0 - C_1 H \right) \tag{16}$$

If we differentiate (16) with respect to t, replace $\dot{\lambda}$ by the right-hand side of (15), and cancel out the e^{-rt}, we obtain (after rearranging terms)

$$\frac{1}{k} \dot{W} - C_1 \dot{H} = \frac{\alpha - 1}{AS} C_1 W + \frac{r}{k} W - \frac{rg}{k} - rC_0 - rC_1 H$$

Substituting the right-hand side of (12) for $\dot{H}$ in the above result and rearranging give

$$\dot{W} = rW - rkC_1 H + \left(\frac{kC_1 R}{AS} - rg - rkC_0 \right) \tag{17}$$

Equations (17) and (12) constitute a system of two differential equations in two unknowns that must be solved simultaneously. To reduce the amount of writing, we present the system of differential equations in compact notations as follows:

$$\dot{H} = mW + M \tag{18}$$

$$\dot{W} = rW - nH + N \tag{19}$$

where

$$m = \frac{\alpha - 1}{AS}$$

$$M = \frac{R}{AS}$$

$$n = rkC_1 \tag{20}$$

$$N = \frac{kC_1 R}{AS} - rg - rkC_0$$

The homogeneous system is

$$\dot{H} = mW \tag{21}$$

$$\dot{W} = rW - nH \tag{22}$$

Differentiating (22) with respect to time gives

$$\ddot{W} = r\dot{W} - n\dot{H}$$

Substituting the right-hand side of (21) for $\dot{H}$ above gives a second-order differential equation:

$$\ddot{W} - r\dot{W} + nmW = 0$$

The solution of the above differential equation is

$$W(t) = Ae^{x_1 t} + Be^{x_2 t} \tag{23}$$

where A and B are arbitrary constants (A should not be confused with AS) and X_1 and X_2 are the roots of the polynomial $x^2 - rx + nm = 0$. Substituting the right-hand side of (23) for W in (21) and integrating give the solution for $H(t)$ as follows:

$$H(t) = m \int^t W \, d\beta = \frac{mA}{x_1} e^{x_1 t} + \frac{mB}{x_2} e^{x_2 t} \tag{24}$$

By setting $\dot{H} = 0$, $\dot{W} = 0$ in (21) and (22), respectively, we obtain the additional information needed for the inhomogeneous solutions. These solutions are

$$H(t) = \frac{mA}{x_1} e^{x_1 t} + \frac{mB}{x_2} e^{x_2 t} + \frac{N - r(M/m)}{n} \tag{25}$$

$$W(t) = Ae^{x_1 t} + Be^{x_2 t} - M/m \tag{26}$$

By setting $t = 0$ in (25) and recalling that $H(0) = H_0$ we obtain an expression for the arbitrary coefficient B in terms of A:

$$B = \frac{x_2}{m} \left(H_0 - \frac{N - r(M/m)}{n} - \frac{mA}{x_1} \right) \tag{27}$$

The analysis of the magnitudes of x_1 and x_2 is postponed to a later point. For the time being we just mention that since $-4nm$ is positive ($-nm = rkC_1(\alpha - 1)/(AS)$ as represented in (20); k in (1) is negative; $C_1 = -C_1'$; hence it is negative; α is the return flow coefficient; since $\alpha < 1$, $\alpha - 1 < 0$; AS is positive), $x_1 > r$ and $x_2 < 0$. The transversality condition states that $\lambda(t) \to 0$ as $t \to \infty$. Or we can utilize (15) and write

$$\lambda(t) = \frac{AS}{\alpha - 1} e^{-rt} \left[\frac{1}{k} W(t) - \frac{g}{k} - C_0 - C_1 H(t) \right] \to 0$$

as $t \to \infty$. Substituting the right-hand sides of (25) and (26) for $H(t)$ and $W(t)$ in the above equation, we realize that the transversality condition can be satisfied only if $A = 0$. The solution of the optimal control system as given in (11) and (12) is, accordingly,

$$H(t) = \frac{mB}{x_2} e^{x_2 t} + \frac{N - r(M/m)}{n} \tag{28}$$

$$W(t) = Be^{x_2 t} - M/m \tag{29}$$

where B is given by (27), with $A = 0$.

COMPARING OPTIMAL CONTROL WITH COMPETITION

Now it remains for us to compare the results of the solutions to the optimal control problem as represented by (28) and (29) with the competition (no control) results as represented by (8) and (9).

We begin by analyzing the root x_2:

$$x_2 = \frac{r - (r^2 - 4nm)^{1/2}}{2}$$

and utilizing (20), we obtain

$$r^2 - 4nm = \left(r - 2kC_1 \frac{\alpha - 1}{AS} \right)^2 - 4\gamma^2$$

where $\gamma = kC_1(\alpha - 1)/AS$. Consequently, if γ is sufficiently small, then the term $4\gamma^2$ can be ignored and

$$x_2 \cong \frac{kC_1(\alpha - 1)}{AS} \tag{30}$$

We then can conclude that $x_2 t$, the exponent of e in the optimal equations (28) and (29), is approximately equal to the exponent of e in the competitive equations.

If we now consider the remaining coefficients in the expressions for $H(t)$ and $W(t)$ for the control and no-control case, we obtain the following expressions:

$$\frac{N - r(M/m)}{n} - \left[\frac{-(\alpha - 1)(g + kC_0) - R}{(\alpha - 1)kC_1} \right] = \frac{R}{rAS}$$

$$\frac{mB}{x_2} - \left\{ \frac{R + (\alpha - 1)(g + kC_0) + [(\alpha - 1)kC_1]H_0}{(\alpha - 1)\, kC_1} \right\} = -\frac{R}{rAS}$$

$$\frac{M}{m} - \left(\frac{-R}{\alpha - 1} \right) = 0$$

$$B - \left[\frac{R}{\alpha - 1} + g + kC_0 + kC_1 H_0 \right] \Big/ kC_1 = -\frac{R}{rAS}$$

TABLE 1. Numerical Estimates of Hydrologic and Economic Parameters Pertaining to the Pecos Basin, New Mexico

Symbol	Description	Numerical Estimates for the Pecos Basin (New Mexico)
g	intercept of the demand-for-water function	470,365 ac ft/yr
k	decrease in demand for water per $1 increase in price	−3,259 ac ft/yr
C_0	intercept of the cost of pumping function*	125 dollars/ac ft
C_1	increase in pumping cost per acre-foot per 1-foot decline in the water table	−0.035 dollars/ac ft per foot of lift
$\alpha - 1$	return flow coefficient minus unity	−0.73 pure number
AS	area of aquifer times storativity	135,000 ac ft/yr
R	natural recharge	173,000 ac ft/yr
H_0	initial water table elevation	3,400 feet above sea level

*$C_0 = C_0' + C_1' S_L$. Hence it should not be interpreted as a fixed cash.

We have used the approximate value of x_2 given by (30) above.

The above implies that if the storage of the aquifer is relatively large (AS is relatively large), then the value of γ will be small, its square will be negligible, and the rate of discount r will practically vanish from the exponent $(x_2 t)$ in the optimal control system. The exponents of the optimal control equations (equations (28) and (29)) will be practically identical with the exponents of the competition equations ((8) and (9)).

The coefficients of the water table H equations differ by the term $R/(rAS)$. We predict that, again, if the storage of the aquifer is relatively large, the two systems would be very close, in fact, identical for all practical purposes.

Finally, the coefficient B in the control equation (29) differs by $-kC_1R/(rAS)$ from its counterpart in (9). This term may be viewed as an adjustor for the demand function for water as given in (1), namely, $W = g + kP$. In other words, if instead of g the intercept of the demand for water were $g - kC_1R/(rAS)$, pumping under competition would have been optimal. We predict, however, that if AS is relatively large, no adjustment is necessary.

An Empirical Illustration

Economics and hydrologic parameters as given by *Gisser and Mercado* [1972] are displayed in Table 1. The results of calculating $H(t)$ and $W(t)$ equations are (for $r = 0.1$)

Competition (no controls)

$$H(t) = 1525 + 1875 \cdot \exp(-0.000617) \cdot t$$

$$W(t) = 237{,}000 + 211{,}056 \cdot \exp(-0.000617) \cdot t$$

Optimal control

$$H(t) = 1538 + 1862 \cdot \exp(-0.000613) \cdot t$$

$$W(t) = 237{,}000 + 211{,}056 \cdot \exp(-0.000613) \cdot t$$

Notice that since $-4\gamma^2$ (equation (30)) is minute, the exponents in the optimal control results are almost identical with the exponents of the competition result. From examining γ in (30) we conclude that for aquifers with relatively large storage capacity AS, this result should hold.

The only significant difference arises between B in (29) and the comparable coefficient in (9). But again, if the storage capacity of the aquifer is large, the difference, which is roughly $kC_1R/(rAS)$, must be relatively very small. The reason for this is that an expression which includes the element R/AS must be significantly smaller than the expression including the element $R/(\alpha - 1)$. It is interesting to note that in both cases the present value of future income streams is $\$3.1 \times 10^8$.

Conclusion

An analytical comparison between a free market behavior (no control) and optimal control was carried out, and it was shown that if the storage capacity of the aquifer AS is relatively large, then the two strategies perform equally well. In the case of the Pecos Basin (New Mexico), which we simplified in order to conform to the mathematical analysis, the results were almost identical.

The conclusion that we reach is very simple. At this stage the economic profession would benefit more from estimating economic and hydrologic parameters than from further discussing optimal control schemes for groundwater management. Our analytical results, tested by empirical estimates of parameters in the Pecos River Basin, give rise to a prediction that temporal optimal control of groundwater would not enhance the welfare of farmers compared with a strategy of free markets. Two points, however, must be stressed here. First, as we stated at the outset, exclusiveness is achieved by assuming that only land overlying the aquifer can be irrigated. What it means is that a farmer cannot stretch a pipe outside the aquifer and irrigate. Second, if the aquifer has a bottom, a steady state solution may not be reached. In that case either junior rights must be called, or alternatively, if all water rights are of the same vintage, water rights should be restricted. This possibility was analyzed by *Gisser and Mercado* [1973]. Third, our model is deterministic; we know that the natural recharge is a stochastic variable. We opted for a deterministic model because the natural recharge is small in relation to the storage capacity of the aquifer.

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